<u>RTCA Special Committee 186, Working Group 3</u>

<u>ADS-B 1090 MOPS, Revision A</u>

<u>Meeting #15</u>

Further Analysis of the 1090 MHz ATCRBS Fruit Arrival Time Data

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SUMMARY
This is a preliminary comparison of the Negative Binomial Distribution (NBD), expressed in non-customary terms of mean and variance, applied to the 1090 MHz ATCRBS Fruit arrival time data reported in RTCA SC-186 Working Group-3 Working Papers WP-12-14, WP-13-05, WP-13-14, WP-14-12, and WP-14-15.

Negative Binomial Distribution (NBD)

INTRODUCTION

This is a preliminary comparison of the negative binomial distribution (NBD), expressed in non-customary terms of mean and variance, applied to the 1090 MHz ATCRBS Fruit arrival time data reported in SC-186 WG-3 working papers WP-12-14, WP-13-05, WP-13-14, WP-14-12, and WP-14-15.

BACKGROUND

A while ago (July 1996) I did a Mark XII IFF/SSR Mode S RF link reliability and system/target capacity study for the Navy. In it I compared results using the binomial, negative binomial, and their limiting form, the Poisson, probability "laws'. I noted "the binomial and negative binomial multi-occurrence probability "laws" may be derived from parent distributions which manifest either a repulsive or attractive (e.g., bunching) nature, respectively, between elements. There is some theoretical rational for thinking the negative binomial probability may better represent signals-in-space but, as will be seen, this can only be operationally checked in situations where the a priori probabilities are not small, i.e., both wide beamwidths and high PRF's not typically employed in IFF/SSR systems."

Hindsight is 20/20, i.e., 20th century statistical physics tells us that odd-spin "particles", such as neutrons, obey the Pauli exclusion principle and Fermi-Dirac (F-D) statistics with an equilibrium binomial distribution (BD); and even-spin "particles", such as photons (EM waves), obey no exclusion principle and Bose-Einstein (B-E) statistics with an equilibrium negative binomial distribution (NBD). The limiting form for both of these statistics is Maxwell-Boltzmann (M-B) statistics with an equilibrium Poisson distribution (PD). It is noted that the discrete distributions have their continuous analogues, i.e., the normal distribution is the continuous analogue of the BD and PD, and the gamma distribution is the continuous analogue of the NBD. In general, because a system is made up of different types of "particles", predicting a distribution is not as easy as it sounds (F-D?), or looks (B-E?). Whether the observed data is a manifestation of this fundamental phenomena and/or just the fact, for example that scheduled interrogations are not random, or that the variances reflect range differences, or some other reason, is not clear at this time.

In the study I considered 10 interrogators and 100 aircraft and compared all three probability laws for a narrow-beam (3.6 deg)/low-PRF (250/sec) case with average transponder transaction periods of 300 microseconds and decode intervals of 128 microseconds, and a wide-beam (36 deg)/high-PRF (2500/sec) case with average transponder transaction periods of 200 microseconds and decode intervals of 50 microseconds. I calculated the probabilities for transponder access (with no capture interrogation overlaps) and occupancy (capture interrogations were used to generate fruit), and interrogator reception (with no fruit overlaps, although a case was considered with a 38% contribution when there was one ATCRBS fruit overlap) and fruit false alarms. The probabilities of transponder access, and interrogator reception (aperture = 2*beamwidth), were used to determine binomial, Poisson, and negative binomial round reliabilities, which for the narrow-beam/low-PRF case were all .9862 (to four significant figures), and for the wide-beam/high-PRF case were .1218, .1326, .1442, respectively. Also calculated, for a given round reliability and fruit false alarm probability, were the detection probability, and the probability of identification for a given number of interrogations and reply threshold.

So, with low performance detection, for higher a priori probabilities (larger beamwidths and/or PRF's), if the EM signals-in-space distribution is negative binomial rather than Poisson, then theory suggests that one gets better than expected results for the round reliability (.1442-.1326/.1326 ~ 9 % improvement, 3 % on the uplink, 6 % on the downlink, link differences related to aperture = 2*beamwidth). However, with higher performance detection, this may not be the case. **(Did WP-14-15 answer the question, at least as far as airborne 1090 MHz ADS-B is concerned?)** It is noted that if the higher performance detection criteria (% detection with each number of overlaps) is given, then the analysis can be redone.

In the current analysis, I have first used the probability distributions expressed in terms of their mean and variance, rather than in the more customary terms of number and a priori probability, because it seems that the BD and NBD are uniquely determined by these two parameters, in the same way that the PD is uniquely determined by its mean. After the comparison with data, then I briefly discuss the question of a priori probabilities, which I may go into in more detail in another paper if there is interest in using it to make predictions. There seems to be a lot of ambiguity and confusion about the NBD, when it is expressed in terms of number and a priori probability, even though it has been around and used for a long time (almost 300 years).

MEAN & VARIANCE

This is a summary of the PD, BD, and NBD, expressed in terms of mean (m) and variance (v), for the random variable z. These expressions were derived from the usual distributions given in terms of number and a priori probability, presented after the Preliminary Results

The PD for dispersed data, where the mean (m) of z is equal to the variance (v) of z (i.e., m=v), becomes:

$P(z;m) = ((m^{z})/z!)(\exp(-m))$
$P(0;m) = \exp(-m)$
$P(1;m) = m(\exp(-m))$
$P(2;m) = ((m^{2})/2)(\exp(-m))$

The BD for under-dispersed data, where the mean (m) of z exceeds the variance (v) of z (i.e., m>v), and z is less than or equal to r, where $r = (m^{2})/(m-v)$, becomes:

$B(z;m,v) = ((r(r-1)(r-2)\ldots(r-z+1))/z!)((1-(v/m))^{z})((v/m)^{(r-z)})$
$B(0;m,v) = (v/m)^{r}$
$B(1;m,v) = m((v/m)^{(r-1)})$
$B(2;m,v) = ((r(r-1))/2)((1-(v/m))^{2})((v/m)^{(r-2)})$

Limit of BD: $r(1-(v/m)) = ((m^{2})/(m-v))(1-(v/m)) = m$, implies $(v/m) = 1-(m/r)$. Therefore, in the limit as v approaches m, r approaches infinity, B(z,m,v) approaches $((m^{z})/z!)((v/m)^{(r-z)})$, which approaches $((m^{z})/z!)((1-(m/r))^{r})$, which approaches $((m^{z})/z!)(\exp(-m))$, which is the PD.

The NBD for over-dispersed data, where the variance (v) of z exceeds the mean (m) of z (i.e., v>m), where $s = (m^{2})/(v-m)$, becomes:

$NB(z;m,v) = ((s(s+1)(s+2)\ldots(s+z-1))/z!)((m/v)^{s})((1-(m/v))^{z})$
$NB(0;m,v) = (m/v)^{s}$
$NB(1;m,v) = m((m/v)^{(s+1)})$
$NB(2;m,v) = ((s(s+1))/2)((m/v)^{s})((1-(m/v))^{2})$

Limit of NBD: $s(1-(m/v)) = ((m^{2})/(v-m))(1-(m/v)) = (m^{2})/v$, implies $(m/v) = 1/(1+(m/s))$. Therefore, in the limit as v approaches m, s approaches infinity, NB(z,m,v) approaches $((m^{z})/z!)((m/v)^{(s+z)})$, which approaches $((m^{z})/z!)(1/(1+(m/s))^{s})$, which approaches $((m^{z})/z!)(\exp(-m))$, which again is the PD.

Notes:

1. *For the PD and NBD, z = 0,1,2,…*
2. *The BD and NBD were derived for non-integer values of r and s, respectively, by converting factorials to gamma functions in the original expressions.*

PRELIMINARY RESULTS

The LA airborne data, only for occupancy numbers zero, one, and two, were compared to the calculated values for the NBD, using the provided means and variances, and for the PD, using the provided mean. The results below (Q explained in the next section) appear to indicate that the NBD fits the data better than the PD. A careful analysis for all the occupancy numbers is required before any final conclusion may be drawn.

LA data (Higher Threshold):

m = 1.6, v = 3.3	f(0)	f(1)	f(2)
Measured	.32	.28	.18
NBD(s=1.51,Q=0.48)	.34	.26	.17
Poisson	.20	.32	.26

LA data (Normal Threshold, -84 dBm at antenna):

m = 1.96, v = 4.4	f(0)	f(1)	f(2)
Measured	.24	.28	.21
NBD(s=1.57,Q=.45)	.28	.24	.17
Poisson	.14	.28	.27

LA data (Lower Threshold):

m = 2.8, v = 6.6	f(0)	f(1)	f(2)
Measured	.13	.22	.21
NBD(s=2.06,Q=.42)	.17	.20	.18
Poisson	.06	.17	.24

Note: The conventional method of comparison between measured and predicted BD or NBD is to do a regression analysis using as fitting parameters a number like r or s, and an a priori probability like P or Q which is described in the next section.

NUMBER & A PRIORI PROBABILITY

This is a summary of the PD, BD, and NBD, for the random variable z, which will be either the number x of arrivals (successes) in a time subinterval, or the number y of non-arrivals (failures) in the same time subinterval. Let X (= $x+y$) be the number of elements (trials) in a time interval, and let P = a priori single element arrival (success) probability = $1-Q$, where Q is the a priori single element non-arrival (failure) probability.

For the PD: If $m = v = XP$, then one obtains the usual form of the PD for success:

$$P(x;XP) = (((XP)^x)/x!)(\exp(-(XP))).$$

For the BD: If $m = XP$ and $v = XPQ$, then $r = X$ (=1,2,…) and $(v/m) = Q$, and one obtains the usual form of the BD:

$$B(x;X,P) = ((X(X-1)(X-2)\ldots(X-x+1))/x!)(P^x)((1-P)^{(X-x)})$$
$$B(x;X,P) = (X!/(x!(X-x)!))(P^x)(Q^{(X-x)}) = B(y;X,Q)$$

For the NBD: If $m = xQ/P$ and $v = xQ/(P^2)$, then $s = x$ (=1,2,…) and $(m/v) = P$, and one obtains the “common” form of the NBD for failure:

$$NB(y;x,P) = ((x(x+1)(x+2)\ldots(x+y-1))/y!)(P^x)((1-P)^y)$$
$$NB(y;x,P) = ((x+y-1)!/(y!(x-1)!))(P^x)((1-P)^y) = P * B(x-1;X-1,P)$$

Or, if $m = yP/Q$ and $v = yP/(Q^2)$, then $s = y$ (=1,2,…) and $(m/v) = Q$, and one obtains an alternate form of the NBD for success:

$$NB(x;y,Q) = ((y(y+1)(y+2)\ldots(y+x-1))/x!)(Q^y)((1-Q)^x)$$
$$NB(x;y,Q) = ((x+y-1)!/(x!(y-1)!))(Q^y)((1-Q)^x) = Q * B(y-1;X-1,Q)$$

I have included the values of the parameters $y = s = (m^2)/(v-m)$ and $Q = (m/v)$, for the alternate form of the NBD for success (arrivals), in the Preliminary Results tables. As the threshold decreases, Q decreases as maybe one might expect for an a priori non-arrival probability, and y increases even though it represents failures. If $y = XQ$ = the expected number of non-arrivals (failures), then $m = XP$ (which is the same as the other two distributions) and $v = XP/Q$. As the threshold is decreased, X increases faster than Q decreases.

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Arrival Process Distributions

Process	Number/ Unit Time	Interarrival Time	Arrival Time
Bernoulli (F-D)	Binomial	Geometric	Negative Binomial
Poisson (M-B)	Poisson	Exponential	Gamma
Pascal (B-E)	Negative Binomial	X	

Interarrival Distribution

f(x) x=no./t	f(0)=1-P(t) Mean=at	f(t)=dP(t)/dt Continuous	f(n) Discrete
Binomial	(1-at/n)^n	a(1-at/n)^(n-1) (Beta)	a(1-a)^(n-1) n=1,2,… Geometric
Poisson	exp(-at)	a(exp(-at)) Exponential	
Negative Binomial	1/(1+at/n)^n	a/(1+at/n)^(n+1) (Pareto)	a/(1+a)^(n+1) n=0,1,2,… X

Interarrival Time

Interarrival Time	Mean	Variance	Variance/ Mean
Geometric	1/a	(1-a)/(a^2)	(1-a)/a
Exponential	1/a	1/(a^2)	1/a
X	1/a	(1+a)/(a^2)	(1+a)/a